Lecture 8: The Tangent Plane

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Math 142:
Differential Geometry

The Tangent Plane

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Proposition

Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3,$$

coincides with the set of tangent vectors to S and $\mathbf{x}(q)$.

The Tangent Plane

Proof.

□

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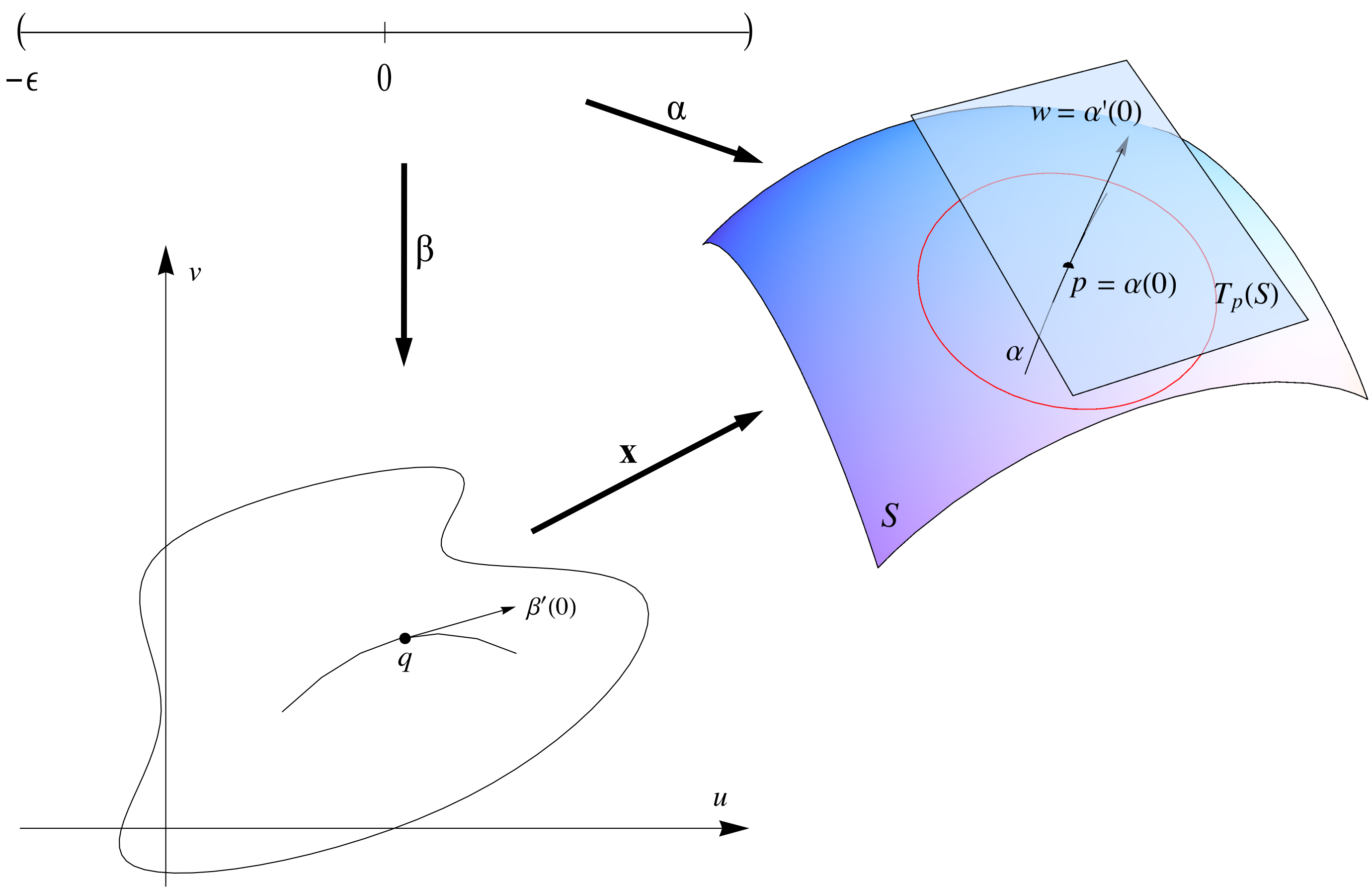
By the above proposition, the plane $d\mathbf{x}_q(\mathbb{R}^2)$, which passes through $\mathbf{x}(q) = p$, does not depend on the parametrization $\mathbf{x}$. This plane will be called the *tangent plane* to S at p and will be denoted $T_p(S)$.



−ϵ
0
α
β
v
u
q
β′(0)
x
w = α'(0)
p = α(0)
$T_p(S)$
α
S

The Tangent Plane

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3. Normal Vector $N(p)$ of $T_p(S)$:

By fixing a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ at $p \in S$, we can make a definite choice of a unit normal vector at each point $q \in \mathbf{x}(U)$ by the rule

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_p}{\|\mathbf{x}_u \wedge \mathbf{x}_p\|}(q).$$

Thus, we obtain a differentiable map $N : \mathbf{x}(U) \to \mathbb{R}^3$.

The Differential of a Map

Moving Between Surfaces

With the notion of a tangent plane, we can talk about the differential of a (differentiable) map between surfaces. Let S_1 and S_2 be two regular surfaces and let $\varphi : V \subset S_1 \to S_2$ be a differentiable mapping of an open set V of S_1 into S_2. If $p \in V$, we know that every tangent vector $w \in T_p(S_1)$ is the velocity vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\epsilon, \epsilon) \to V$ with $\alpha(0) = p$. The curve $\beta = \varphi \circ \alpha$ is such that $\beta(0) = \varphi(p)$, and therefore $\beta'(0)$ is a vector of $T_{\varphi(p)}(S_2)$.

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Proposition

In the discussion above, given w, the vector $\beta'(0)$ does not depend on the choice of α. The map $d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ defined by $d\varphi_p(w) = \beta'(0)$ is linear.

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Definition

The linear map $d\varphi_p$ is called the *differential* of φ at $p \in S_1$. In a similar way we define the differential of a (differentiable) function $f : U \subset S \to \mathbb{R}$ at $p \in U$ as a linear map $df_p : T_p(S) \to \mathbb{R}$.

Example 1

Let $v \in \mathbb{R}^3$ be a unit vector and let $h : S \to \mathbb{R}$, $h(p) = v \cdot p$, $p \in S$, be the height function. To compute $dh_p(w)$, $w \in T_p(S)$,

Helpful Hints

Key Techniques on Using

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- Tangent Plane
- Inverse Function Theorem

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- Try to set your coordinates smartly to use the tangent plane
- Try to set up certain functional relationships so that you can use the Inverse Function Theorem

Examples

Proposition

If S_1 and S_2 are regular surfaces and $\varphi : U \subset S_1 \to S_2$ is a differentiable mapping of an open set $U \subset S_1$ such that the differential $d\varphi_p$ of φ at $p \in U$ is an isomorphism, then φ is a local diffeomorphism at p.

Examples

Do Carmo, p. 90, #15

Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

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Solution

Without loss of generality, assume that all normals pass through the origin. Let $\mathbf{x}(u, v)$ be a parametrization of S at p. Say $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$. To show that the image of $\mathbf{x}$ is contained in a sphere, we will show that $\|x(u, v)\|^2$ is constant.

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Since all the normals to the surface pass through the origin, we may write $k(u, v)N(u, v) = \mathbf{x}(u, v)$, where $N(u, v)$ is the normal to the surface at the point $\mathbf{x}(u, v)$.

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$$\begin{aligned}\frac{\partial}{\partial u}\|\mathbf{x}(u, v)\|^2 &= \frac{\partial}{\partial u}(x^2(u, v) + y^2(u, v) + z^2(u, v)) \\ &= 2x(u, v)\frac{\partial x}{\partial u} + 2y(u, v)\frac{\partial y}{\partial u} + 2z(u, v)\frac{\partial z}{\partial u} \\ &= 2kN \cdot \mathbf{x}_u = 0.\end{aligned}$$

Examples

Solution (cont'd)

Similarly, $\frac{\partial}{\partial v}\|x(u,v)\|^2 = 2kN \cdot \mathbf{x}_v = 0$. Thus, $\|\mathbf{x}(u,v)\|^2$ is constant, so $\mathbf{x}(u,v)$ is contained in a sphere. By the connectedness of S, S must lie on the same sphere. $\square$

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Remark

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(u,v) = \|\mathbf{x}(u,v)\|^2$. Then $df_p = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right) = (0,0)$ by Proposition 9, so f is constant on U.

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Remark

We cannot use a similar method to show #4, p. 23, because if we show that $\|x(t)\|$ is constant, then $\mathbf{x}(t)$ lies on a sphere, but this does not imply that $\mathbf{x}(t)$ is contained in a circle.

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Let us set up a coordinate system with the origin at p and with P coinciding with the xy plane. Since S meets P only at p, p must be a critical point of z when we view a neighborhood of p as a graph of $z = f(x, y)$.

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To show that $T_p(S) = P$, it suffices to show that $T_p(S) \subset P$, since $\dim T_p(S) = \dim P = 2$.

Let $v \in T_p(S)$. Then there is some $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$ such that $v = \alpha'(0) = (x'(0), y'(0), z'(0))$. Since $z(0)$ is a critical point of 0, it follows that $z'(0) = 0$. Then $v = (x'(0), y'(0), 0) \in P$. Thus, $T_p(S) \subset P$. □

Something Useful Later On

Say $z = f(x, y)$ and $p = (x_0, y_0)$ is the critical point of the function $z = f(x, y)$ (i.e., $\frac{\partial f}{\partial x}(\mathrm{p}) = \frac{\partial f}{\partial y}(\mathrm{p}) = 0$). Now, using Taylor expansion, we have

$$f(x + x_0, y + y_0) = f(x_0, y_0) + \overbrace{\begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix}}^{0} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$
$$+ \begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial xy}(p) \\ \frac{\partial^2 f}{\partial xy}(p) & \frac{\partial^2 f}{\partial y^2}(p) \end{pmatrix}}_{M} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \cdots,$$

or

$$f(x + x_0, y + y_0) - f(x_0, y_0) = \begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix} M \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

If M is positive definite, then $p(x_0, y_0)$ is a minimum point since $f(x + x_0, y + y_0) > f(x_0, y_0)$ and if M is negative definite, then $p(x_0, y_0)$ is a maximum point.